Government Polytechnic, Ahmedabad

Science and Humanities Department

Unity in Diversity

Engineering Mathematics

Subject Code: 4320002

Unit -04 Differential Equation

[Marks – 12]

Course Outcome (CO d):

Develop the ability to apply differential equations to significant applied problems.

Assignment - 4

Unit - 04 Differential Equation

→ Question Set for 01 Mark:

1. The degree of the differential equation:
$$x\left(\frac{d^2y}{dx^2}\right)^3 + y\left(\frac{dy}{dx}\right)^4 - 7y = 0 \text{ is } \underline{3}$$

Ans d) 3

2. The order of the differential equation:
$$\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{d^2y}{dx^2}\right)^4 + x\sin y = 0 \text{ is } \underline{3}$$

Ans d) 3

(3) The order and degree of the differential equation $\sqrt{\frac{d^2y}{dx^2}} = \sqrt[3]{\frac{dy}{dx}}$ are respectively $\underline{(2,3)}$

Ans $\sqrt{\frac{d^2y}{dx^2}} = \sqrt[3]{\frac{dy}{dx}}$

$$\left(\frac{d^2y}{dx^2}\right)^3 = \left(\frac{dy}{dx}\right)^2 \qquad \therefore \text{order} = 2$$
$$\text{degree} = 3$$

Ans a) 2, 3

(4) The degree of the differential equation
$$\frac{d^2y}{dx^2} + \sin\left(\frac{dy}{dx}\right) + 3y = 0 \text{ is } \underline{\text{Undefined}}$$

Ans: d) Undefined

(5) The integrating factor of the differential equation $\frac{dy}{dx} = y\tan x + e^x$ is $\underline{\cos x}$

Ans. d) $\cos x$

(6) The degree of the homogeneous function $f(x,y) = \frac{x^2+y^2}{x+y}$ is $\underline{4}$

Ans: c) 4

(7) The integrating factor of the differential equation: $x\frac{dy}{dx} = x+y$ is $\underline{-x}$

→ d) $-x$

(8) For solving $\frac{dy}{dx} + Py = Q$ the suitable method is ___

→ b)
Integrating factor method.

(9) The general solution of the differential equation $\frac{dy}{dx} = \frac{y}{x}$

$\int \frac{1}{y}dy = \int \frac{1}{x}dx$

$\log|y| = \log|x| + \log c$

$\boxed{y = cx}$ Ans

→ b) $y = cx$

10. The type of solution of the differential equation.

$\left(x\tan\frac{y}{x} - y\sec^2\frac{y}{x}\right)dx + x\sec^2\left(\frac{y}{x}\right)dy = 0$ is

→ a) Homogeneous ~~fo~~ method

> Question Set for 03 marks:

1. Solve the following Differential equations:

a) $\tan y\,dx + \tan x \sec^2 y\,dy = 0$

Solⁿ: $\tan y\,dx = -\tan x \sec^2 y\,dy$

$\therefore \dfrac{\tan y}{\tan x \cdot \tan y} = -\dfrac{\tan x \cdot \sec^2 y}{\tan x \cdot \tan y}\,dy$

(Dividing both the sides $\tan x \cdot \tan y$)

$\therefore \dfrac{1}{\tan x} = -\dfrac{\sec^2 y}{\tan y}\,dy$

$\therefore \dfrac{1}{\tan x}\,dx + \dfrac{\sec^2 y}{\tan y}\,dy = 0$

$\therefore \int \dfrac{1}{\tan x}\,dx + \int \dfrac{\sec^2 y}{\tan y}\,dy = 0$

$\therefore \int \cot x\,dx + \int \dfrac{\sec^2 y}{\tan y}\,dy = 0$

$\therefore \log|\sin x| + \log|\tan y| = \log|c|$ (where c is arbitrary constant)

$\therefore \boxed{\sin x \cdot \tan y = c}$ Ans

b) $x\frac{dy}{dx} + \cot y = 0$

Soln $x\frac{dy}{dx} + \cot y = 0$ (given question)

$\therefore \frac{1}{\cot y} dy = -\frac{1}{x} dx$

(Integrating both the sides)

$\therefore \int \frac{1}{\cot y} dy = -\int \frac{1}{x} dx$

$\therefore \int \tan y \, dy = -\int \frac{1}{x} dx$

$\therefore \log|\sec y| = -\log|x| + \log|c|$

$\therefore \log|\sec y \cdot x| = \log|c|$

$\therefore \boxed{x \sec y = c}$ Ans

c) $\frac{dy}{dx} = \sin(x+y)$ —(1)

Soln Substituting $x+y=t$

$1 + \frac{dy}{dx} = \frac{dt}{dx}$

$\therefore \frac{dy}{dx} = \frac{dt}{dx} - 1$

$\therefore$ taking $\frac{dy}{dx}$ in equation no (1)

$\therefore \frac{dt}{dx} - 1 = \sin(t)$

$\therefore \frac{dt}{dx} = 1 + \sin t$

$\therefore \frac{1}{1+\sin t} dt = dx$

$\therefore \frac{1}{1+\sin t} \cdot \frac{1-\sin t}{1-\sin t} \cdot dt = dx$

$\therefore \frac{1-\sin t}{1^2 - \sin^2 t} dt = dx$

$\therefore \frac{1-\sin t}{\cos^2 t} dt = dx$

$\therefore (\sec^2 t - \tan t \sec t) dt = dx$

Integrating both the sides

$\therefore \int (\sec^2 t - \tan t \cdot \sec t) dt = \int 1 dx$

$\therefore \tan t - \sec t = x + c$

Where c is arbitrary constant

$\therefore$

$\boxed{\tan(x+y) - \sec(x+y) = x + c}$

Ans

$(\because x + y = t)$

d) $\frac{dy}{dx} + y = \cos x - \sin x$

Solⁿ Comparing with $\frac{dy}{dx} + Py = Q$

Where $P = 1$ and $Q = \cos x - \sin x$

$\therefore$ Which are function of x only

Given Diff. equ. is Linear Diff equⁿ.

Integrating Factor $= e^{\int P dx}$

$= e^{\int 1 dx}$

$= e^{x}$ $\quad (\because \int 1 dx = x)$

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$\therefore$ Solution is

$\therefore \quad y(I.F) = \int Q\,(I.F)\,dx + c$

$y(e^x) = \int \cos x - \sin x\,(e^x)\,dx + c$

$\therefore \quad e^x y = \int e^x \cos x\,dx - e^x \sin x\,dx + c$

$\therefore$ ~~$e^x y =$~~

Let $u = \cos x$

$v = e^x$

$\therefore \quad \int e^x \cos x\,dx = e^x \cos x - [\int -\sin x\, e^x\,dx]$

Now:

$e^x y = e^x \cos x - \int[-\sin x\, e^x\,dx - e^x \sin x\,dx + c$

$e^x y = e^x \cos x + \int \cancel{\sin x\, e^x}\,dx - e^x \cancel{\sin x}\,dx + c$

$\boxed{e^x y = e^x \cos x + c}$ Ans

Where c is arbitrary constant

e) $x\dfrac{dy}{dx} + \dfrac{y}{x} = x^2$

Ans

$x\dfrac{dy}{dx} = \dfrac{x^3 - y}{x}$

$\therefore \dfrac{dy}{dx} = x^3 - y$

Now $\dfrac{dy}{dx} + \dfrac{y}{x^2} = x$

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e. $\frac{dy}{dx} + \frac{y}{x} = x^2$

Sol^n Comparing with $\frac{dy}{dx} + Py = Q$

where $P = \frac{1}{x}$ and $Q = x^2$

which are function of x only

Given Diff. equ^n is Linear diff equ^n

Integrating factor (I.F) $= e^{\int P dx}$

$$= e^{\int \frac{1}{x} dx}$$

$$= e^{\log x}$$

$$= x$$

Solution is : $y(I.F) = \int Q (I.F) dx + c$

$y(x) = \int x^2 (x) dx$

$\therefore yx = \int x^3 dx$

$\therefore \boxed{yx = \frac{x^4}{4} + c}$ Ans

(f) $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$

Sol^n $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$

$\therefore \frac{1}{\sqrt{1-y^2}} dy = \frac{1}{\sqrt{1-x^2}} dx$

Intigrating both the sides.

$$\therefore \int \frac{1}{\sqrt{1-y^2}}\,dy = \int \frac{1}{\sqrt{1-x^2}}\,dx$$

$$\therefore \sin^{-1} y = \sin^{-1} x + c$$

$$\therefore \boxed{\sin^{-1} y - \sin^{-1} x = c} \text{ Ans}$$

9) $\frac{dy}{dx} + y\tan x = \sec x.$

Sol^n Comparing with $\frac{dy}{dx} + Py = Q$

where $P = \tan x$ and $Q = \sec x$
which is functions of x only
Given Diff equ^n is linear Diff equ^n
Integrating Factor (I.F) $= e^{\int P dx}$

$$= e^{\int \tan x\,dx}$$

$$= e^{\log_e \sec x}$$

$$I.F = \sec x$$

Solution is

$$y\,(I.F) = \int Q.(I.F)\,dx + c$$

$$y\,(\sec x) = \int \sec x(\sec x)\,dx + c$$

$$y\sec x = \int \sec^2 x\,dx + c$$

$$\boxed{y\sec x = \tan x + c} \text{ Ans}$$

h) $\frac{dy}{dx} = 1 + x + y + xy.$

Sol^n $\frac{dy}{dx} = 1 + x + y(1+x)$

$\frac{dy}{dx} = (1+x)(1+y)$

$\therefore \int \frac{1}{1+y} dy = \int (1+x) dx + c$ (Integrating both the sides)

$\therefore \log|1+y| = \int 1 dx + \int x dx + c$

$\therefore \boxed{\log|1+y| = x + \frac{x^2}{2} + c}$ Ans

x Question set for 04 marks:

1. Solve: $\frac{dy}{dx} = y - 1$

Sol^n $\frac{dy}{dx} = y - 1$

$\therefore \frac{1}{(y-1)} dy = dx$

$\therefore$ Integrating both the sides

$\therefore \int \frac{1}{(y-1)} dy = \int 1 dx$

$\therefore \log|y-1| = x$

$\therefore \boxed{x = \log|y-1|}$ Ans

(2) Solve: $(1+x^2)\frac{dy}{dx} = y$

Sol^n $(1+x^2)\frac{dy}{dx} = y$

$\therefore \frac{1}{y} dy = \frac{dx}{(1+x^2)}$

Integrating both the sides.

$\therefore \int \frac{1}{y} dy = \int \frac{1}{1+x^2} dx$

$\therefore \boxed{\log|y| = \tan^{-1}x + c}$ Ans

Meditation is the best ode of worship.

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(3) Solve : $x(1+y^2)dx = y(1+x^2)dy$

Sol.n $x(1+y^2)dx = y(1+x^2)dy$

$\therefore \frac{x}{1+x^2}dx = \frac{y}{(1+y^2)}dy$

$\therefore \frac{1}{2}\left(\frac{2x}{1+x^2}\right)dx = \frac{1}{2}\left(\frac{2y}{1+y^2}\right)dy$

$\therefore$ Integrating both the sides

$\therefore \frac{1}{2}\log|1+x^2| = \frac{1}{2}\log|1+y^2| + \log c$

$\left(\because \frac{f'(x)}{f(x)} = \log|f(x)|+c\right)$

$\therefore \frac{1}{2}\log|1+x^2| - \frac{1}{2}\log|1+y^2| = \log c$

$\therefore \frac{1}{2}\left[\log|1+x^2| - \log|1+y^2|\right] = \log c$

$\therefore \frac{1}{2}\left[\log\left|\frac{1+x^2}{1+y^2}\right|\right. = \log c$

$\therefore \cancel{\log}\left|\frac{1+x^2}{1+y^2}\right|^{1/2} = \cancel{\log} c$

$\therefore \boxed{\left(\frac{1+x^2}{1+y^2}\right)^{1/2} = c}$ Ans

(4) Solve $\sec^2 x \cdot \tan y \, dx + \sec^2 y \cdot \tan x \, dy = 0$

Sol^n $\sec^2 x \cdot \tan y \, dx + \sec^2 y \cdot \tan x \, dy = 0$

Dividing both the sides of $\tan x \cdot \tan y$

$$\therefore \frac{\sec^2 x \cdot \tan y \, dx}{\tan x \cdot \tan y} = -\frac{\sec^2 y \tan x}{\tan x \cdot \tan y} dy$$

$$\therefore \frac{\sec^2 x}{\tan x} dx = -\frac{\sec^2 y}{\tan y} dy$$

$\therefore$ Integrating both the sides.

$$\therefore \int \frac{\sec^2 x}{\tan x} dx = -\int \frac{\sec^2 y}{\tan y} dy$$

$$\therefore \log|\tan x| = -\log|\tan y| + \log c$$

$$\therefore \log|\tan x| + \log|\tan y| = \log c$$

$$\therefore \cancel{\log}|\tan x \cdot \tan y| = \cancel{\log} c$$

$$\therefore \boxed{\tan x \cdot \tan y = c} \text{ Ans}$$

(5) Solve $\frac{dy}{dx} = \frac{y + \sqrt{x^2+y^2}}{x}$.

Soln $\frac{dy}{dx} = \frac{y + \sqrt{x^2+y^2}}{x}$

$\frac{dy}{dx} = \frac{y}{x} + \frac{\sqrt{x^2+y^2}}{x}$

$\frac{dy}{dx} = \frac{y}{x} + \sqrt{\frac{x^2+y^2}{x^2}}$

$\frac{dy}{dx} = \frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}}$

Let $\frac{y}{x} = v$

$y = vx$

$\frac{dy}{dx} = v\frac{dx}{dx} + x\frac{dv}{dx}$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

Now $v + x\frac{dv}{dx} = v + \sqrt{1-v^2}$

$x\frac{dv}{dx} = \sqrt{1-v^2}$

Integrating both the sides.

$\therefore \int \frac{1}{\sqrt{1-v^2}} dv = \int \frac{1}{x} dx$

$\therefore \sin^{-1} v = \log |x| + c$

$\left(\because \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x,\ \int \frac{1}{x} dx = \log x\right)$

$\therefore \boxed{\sin^{-1}\left(\frac{y}{x}\right) = \log x + c}$ Ans

(6) Solve $\frac{dy}{dx} = \frac{y}{x} + x\sin\left(\frac{y}{x}\right)$.

Soln Let $\frac{y}{x} = t$

$y = xt$

$\therefore \frac{dy}{dx} = x \cdot \frac{dt}{dx} + t\frac{dx}{dx}$

$\frac{dy}{dx} = t + x\frac{dt}{dx}$

Now $\frac{dy}{dx} = t + x\frac{dt}{dx}$

$\therefore \cancel{t} + x\frac{dt}{dx} = \cancel{t} + x\sin t$

$\therefore x\frac{dt}{dx} = x\sin t$

$\therefore \frac{1}{\sin t} dt = 1\, dx$

Integrating both the sides

$\therefore \int \frac{1}{\sin t}\, dt = \int 1\, dx$

$\therefore \int \operatorname{cosec} t\, dt = \int 1\, dx$

$\therefore \log\left|\tan \frac{t}{2}\right| = x + c$

$(\because t = y/x)$

$\therefore \boxed{\log\left|\tan\left(\frac{y}{2x}\right)\right| = x + c}$ Ans

(1) Solve $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$.

$\underline{w^n}$ $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$

$\therefore y^2 = xy\frac{dy}{dx} - x^2\frac{dy}{dx}$

$= \frac{dy}{dx}(xy - x^2)$

$\therefore \frac{dy}{dx} = \frac{y^2}{xy - x^2}$

dividing x^2 numerator and dinomenator

$\therefore \frac{dy}{dx} = \frac{y^2/x^2}{\frac{xy}{x^2} - \frac{x^2}{x^2}}$

$= \frac{y^2/x^2}{y/x - 1}$

$$\frac{dy}{dx} = \frac{(y/x)^2}{y/x - 1}$$

Let $\frac{y}{x} = v$

$y = vx$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore v + x\frac{dv}{dx} = \frac{v^2}{v-1}$

$\therefore x\frac{dv}{dx} = \frac{v^2}{v-1} - v$

$\therefore \frac{x\,dv}{dx} = \frac{v^2 - v^2 + v}{v-1}$

$\therefore x\frac{dv}{dx} = \frac{v}{v-1}$

$\therefore \frac{v-1}{v}\,dv = \frac{1}{x}\,dx$

Integrating both the sides

$\therefore \int \frac{v-1}{v}\,dv = \int \frac{1}{x}\,dx$

$\therefore \int \frac{v}{v} - \frac{1}{v}\,dv = \int \frac{1}{x}\,dx$

$\therefore \int 1 - \frac{1}{v}\,dv = \int \frac{1}{x}\,dx$

$\therefore v - \log|v| = \log x + c$

$$\therefore \boxed{\frac{y}{x} - \log\left(\frac{y}{x}\right) = \log x + c}$$

Ans

(8) Solve $x\frac{dy}{dx} = y + x\cos^2\left(\frac{y}{x}\right)$.

Sol^n $x\frac{dy}{dx} = y + x\cos^2\left(\frac{y}{x}\right)$

$\therefore \frac{dy}{dx} = \frac{y + x\cos^2\left(\frac{y}{x}\right)}{x}$

$\frac{dy}{dx} = \frac{y}{x} + \cos^2\left(\frac{y}{x}\right)$

$\therefore$ Let $\frac{y}{x} = v$

$y = vx$

$\frac{dy}{dx} = v\frac{dx}{dx} + x\frac{dv}{dx}$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

Now

$\cancel{v} + x\frac{dv}{dx} = \cancel{v} + \cos^2 v$

$\therefore x\frac{dv}{dx} = \cos^2 v$

$\therefore \frac{1}{\cos^2 v} dv = \frac{1}{x} dx$

Integrating both the sides.

$\therefore \int \sec^2 v \, dv = \int \frac{1}{x} dx$

$\therefore \tan v = \log|x| + c$

$\therefore \tan\frac{y}{x} = \log|x| + c$

$\therefore \boxed{\tan\frac{y}{x} = \log|x| + c}$

Ans

(9) Find solution of the differential equation using given initial condition.

(a) $\frac{dy}{dx} = (4x+y+1)^2$; $y(0) = 1$

Sol^n $\frac{dy}{dx} = (4x+y+1)^2$ — (1)

Let $4x+y+1 = t$

$\therefore 4(1) + \frac{dy}{dx} + 0 = \frac{dt}{dx}$

$\therefore 4 + \frac{dy}{dx} = \frac{dt}{dx}$

$\therefore \frac{dy}{dx} = \frac{dt}{dx} - 4$

Substituting the value of $\frac{dy}{dx}$ in eq^n (1)

$\frac{dt}{dx} - 4 = t^2$

$\frac{dt}{dx} = t^2 + 4$

$\therefore \frac{1}{t^2+4} dt = dx$

$\therefore \frac{1}{t^2+2^2} dt = dx$

Integrating both the sides

$\therefore \int \frac{1}{t^2+2^2} dt = \int 1 dx$

$\therefore \frac{1}{2} \tan^{-1}\left(\frac{t}{2}\right) = x + c$

$\therefore \tan^{-1}\left(\frac{t}{2}\right) = 2x + c$ — (2)

Result (2) is the general solution of the differential equation.

Now condition (initial) $y(0) = 1$

ie $x = 0, y = 1$

From (2)

$\tan^{-1}\left(\frac{t}{2}\right) = 2x + c$

$\therefore \tan^{-1}\left(\frac{4x+y+1}{2}\right) = 2x + c$

$\therefore \tan^{-1}\left(\frac{4(0)+1+1}{2}\right) = 2(0) + c$

$\therefore \tan^{-1}\left(\frac{2}{2}\right) = 0 + c$

$\therefore \tan^{-1} 1 = c$

$\therefore \frac{\pi}{4} = c$

$c = \pi/4$

$$\boxed{\text{Particular solution: } \tan^{-1}\left(\frac{4x+y+1}{2}\right) = 2x + \pi/4}$$

(b) $\frac{dy}{dx} = y\tan x + e^{x}$; $y(0) = 1$

Sol^n $\frac{dy}{dx} = y\tan x + e^{x}$

Comparing with $\frac{dy}{dx} + Py = Q$

$\therefore \frac{dy}{dx} - y\tan x = e^{x}$

Where $P = -\tan x$ and $Q = e^{x}$

which is function of x only.

Given Diff. eq^n is Linear diff. equ^n.

Integrating factor (I.F) $= e^{\int P\,dx}$

$= e^{\int -\tan x\,dx}$

$= e^{-\log(\sec x)}$ $\left[\sec^{-1} = \frac{1}{\sec x} = \cos x\right]$

I.F $= \cos x$

Solution is

$$y(I.F) = \int Q\,(I.F)\,dx + c$$

$$\therefore y(\cos x) = \int e^x(\cos x)\,dx + c$$

$$\therefore y\cos x = \int e^x \cos x\,dx + c$$

$$\therefore y\cos x = \cos x\, e^x - \int\left[-\sin x \cdot e^x\right]dx + c$$

$$= \cos x\, e^x + \int \sin x\, e^x\,dx + c$$

$$= \cos x\, e^x + \left[\sin x\, e^x - \int \cos x \cdot e^x\,dx\right] + c$$

$$= \cos x\, e^x + \sin x\, e^x - \int \cos x\, e^x\,dx$$

$$I = \cos x\, e^x + \sin x\, e^x - I$$

$$\therefore 2I = e^x(\cos x + \sin x)$$

$$y\cos x \cancel{I} = \frac{e^x}{2}(\cos x + \sin x) + c \quad \text{(i)}$$

Now given initial condition $y(0) = 1$

$\therefore x = 0, \; y = 1$

From(i)

$$1 \cdot \cos 0 = \frac{e^0}{2}(\cos 0 + \sin 0) + c$$

$$1 \cdot 1 = \frac{1}{2}(1 + 0) + c$$

$$1 = \frac{1}{2} + c \qquad \therefore c = 1/2$$

From (i) the particular solution is

$$\boxed{y\cos x = \frac{e^x}{2}(\cos x + \sin x) + \frac{1}{2}}$$ Ans

(c) $\frac{dy}{dx} + 2y\tan x = \sin x$; when $x = \pi/3$ then $y = 0$

solⁿ $\frac{dy}{dx} + 2y\tan x = \sin x$

comparing with $\frac{dy}{dx} + Py = Q$

where $P = 2\tan x$, $Q = \sin x$

which is function of x only

Given diff. equation is Linear Diff equation.

Now

Integrating factor (I.F) $= e^{\int P\,dx}$

$= e^{\int 2\tan x\,dx}$

$= e^{2\int \tan x\,dx}$

$= e^{2\log \sec x}$

$= e^{\log(\sec^2 x)}$

$(I.F) = \sec^2 x$

solution is.

$y(I.F) = \int Q(I.F)\,dx + c$

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$\therefore y(\sec^2 x) = \int \sin x (\sec^2 x)\, dx + c$

$\therefore y\sec^2 x = \int \tan x \cdot \sec x\, dx + c$

$\therefore y\sec^2 x = \sec x + c$ —(i)

$\tan x = \dfrac{\sin x}{\cos x} = \sin x \cdot \sec x$

Now given initial condition

$\therefore x = \pi/3$ then $y = 0$

$\therefore$ from (i)

$\therefore 0 = \sec(\pi/3) + c$

$\therefore 0 = 2 + c$ $\quad (\because \sec \pi/3 = 2)$

$\boxed{c = -2}$

from (i) the particular solution is

$\therefore \boxed{y\sec^2 x = \sec x - 2}$ Ans